The 79th William Lowell Putnam Mathematical Competition
Saturday, December 1, 2018

A1 Find all ordered pairs (a,b) of positive integers for which

$$\frac{1}{a}+\frac{1}{b}=\frac{3}{2018}.$$

A2 Let $S_1,S_2,\dots,S_{2^n-1}$ be the nonempty subsets of $\{1,2,\dots,n\}$ in some order, and let M be the $(2^n-1)\times(2^n-1)$ matrix whose (i,j) entry is

$$m_{ij}=\begin{cases}0 & \text{if } S_i\cap S_j=\emptyset;\\ 1 & \text{otherwise.}\end{cases}$$

Calculate the determinant of M.

A3 Determine the greatest possible value of $\sum_{i=1}^{10}\cos(3x_i)$ for real numbers $x_1,x_2,\dots,x_{10}$ satisfying $\sum_{i=1}^{10}\cos(x_i)=0$.

A4 Let m and n be positive integers with $\gcd(m,n)=1$, and let

$$a_k=\left\lfloor\frac{mk}{n}\right\rfloor-\left\lfloor\frac{m(k-1)}{n}\right\rfloor$$

for $k=1,2,\dots,n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1}gh^{a_2}\cdots gh^{a_n}=e,$$

where e is the identity element. Show that $gh=hg$. (As usual, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

A5 Let $f:\mathbb{R}\to\mathbb{R}$ be an infinitely differentiable function satisfying $f(0)=0$, $f(1)=1$, and $f(x)\geq 0$ for all $x\in\mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x)<0$.

A6 Suppose that A,B,C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\operatorname{area}(\triangle ABC)}{\operatorname{area}(\triangle ABD)}$$

is a rational number.

B1 Let $\mathscr{P}$ be the set of vectors defined by

$$\mathscr{P}=\left\{\begin{pmatrix}a\\b\end{pmatrix}\middle|\, 0\leq a\leq 2, 0\leq b\leq 100, \text{ and } a,b\in\mathbb{Z}\right\}.$$

Find all $\mathbf{v}\in\mathscr{P}$ such that the set $\mathscr{P}\setminus\{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathscr{P}$ can be partitioned into two sets of equal size and equal sum.

B2 Let n be a positive integer, and let $f_n(z)=n+(n-1)z+(n-2)z^2+\cdots+z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z\in\mathbb{C}\colon |z|\leq 1\}$.

B3 Find all positive integers $n<10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides 2^n-1, and $n-2$ divides 2^n-2.

B4 Given a real number a, we define a sequence by $x_0=1$, $x_1=x_2=a$, and $x_{n+1}=2x_nx_{n-1}-x_{n-2}$ for $n\geq 2$. Prove that if $x_n=0$ for some n, then the sequence is periodic.

B5 Let $f=(f_1,f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}-\frac{1}{4}\left(\frac{\partial f_1}{\partial x_2}+\frac{\partial f_2}{\partial x_1}\right)^2>0$$

everywhere. Prove that f is one-to-one.

B6 Let S be the set of sequences of length 2018 whose terms are in the set $\{1,2,3,4,5,6,10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860}\cdot\left(\frac{2018}{2048}\right)^{2018}.$$