A1 Find all ordered pairs \((a,b)\) of positive integers for which
\[
\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}
\]

A2 Let \(S_1,S_2,\ldots,S_{2^n-1}\) be the nonempty subsets of \(\{1,2,\ldots,n\}\) in some order, and let \(M\) be the \((2^n-1)\times(2^n-1)\) matrix whose \((i,j)\) entry is
\[
m_{ij} = \begin{cases} 
0 & \text{if } S_i \cap S_j = \emptyset; \\
1 & \text{otherwise}.
\end{cases}
\]
Calculate the determinant of \(M\).

A3 Determine the greatest possible value of \(\sum_{i=1}^{10} \cos(3x_i)\) for real numbers \(x_1,x_2,\ldots,x_{10}\) satisfying \(\sum_{i=1}^{10} \cos(x_i) = 0\).

A4 Let \(m\) and \(n\) be positive integers with \(\gcd(m,n) = 1\), and let
\[
a_k = \left\lfloor \frac{mk}{n} \right\rfloor - \left\lfloor \frac{m(k-1)}{n} \right\rfloor
\]
for \(k = 1,2,\ldots,n\). Suppose that \(g\) and \(h\) are elements in a group \(G\) and that
\[
gh^{a_1}gh^{a_2}\cdots gh^{a_n} = e,
\]
where \(e\) is the identity element. Show that \(gh = hg\). (As usual, \([x]\) denotes the greatest integer less than or equal to \(x\).)

A5 Let \(f: \mathbb{R} \to \mathbb{R}\) be an infinitely differentiable function satisfying \(f(0) = 0\), \(f(1) = 1\), and \(f(x) \geq 0\) for all \(x \in \mathbb{R}\). Show that there exist a positive integer \(n\) and a real number \(x\) such that \(f^{(n)}(x) < 0\).

A6 Suppose that \(A, B, C,\) and \(D\) are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments \(AB, AC, AD, BC, BD,\) and \(CD\) are rational numbers, then the quotient
\[
\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}
\]
is a rational number.

B1 Let \(\mathcal{P}\) be the set of vectors defined by
\[
\mathcal{P} = \left\{ \left( \frac{a}{b} \right) \middle| 0 \leq a \leq 2, 0 \leq b \leq 100, \text{ and } a,b \in \mathbb{Z} \right\}.
\]
Find all \(v \in \mathcal{P}\) such that the set \(\mathcal{P} \setminus \{v\}\) obtained by omitting vector \(v\) from \(\mathcal{P}\) can be partitioned into two sets of equal size and sum.

B2 Let \(n\) be a positive integer, and let \(f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}\). Prove that \(f_n\) has no roots in the closed unit disk \(\{z \in \mathbb{C} : |z| \leq 1\}\).

B3 Find all positive integers \(n < 10^{100}\) for which simultaneously \(n\) divides \(2^n\), \(n-1\) divides \(2^{n-1}\), and \(n-2\) divides \(2^{n-2}\).

B4 Given a real number \(a\), we define a sequence by \(x_0 = 1\), \(x_1 = x_2 = a\), and \(x_{n+1} = 2x_nx_{n-1} - x_{n-2}\) for \(n \geq 2\). Prove that if \(x_n = 0\) for some \(n\), then the sequence is periodic.

B5 Let \(f = (f_1,f_2)\) be a function from \(\mathbb{R}^2\) to \(\mathbb{R}^2\) with continuous partial derivatives \(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2}\) that are positive everywhere. Suppose that
\[
\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{1}{4} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right)^2 > 0
\]
everywhere. Prove that \(f\) is one-to-one.

B6 Let \(S\) be the set of sequences of length 2018 whose terms are in the set \(\{1,2,3,4,5,6,10\}\) and sum to 3860. Prove that the cardinality of \(S\) is at most
\[
2^{3860} \cdot \left( \frac{2018}{2048} \right)^{2018}.
\]